

Quasineutral limit, dispersion and oscillations for Korteweg type fluids

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Abstract

In the setting of general initial data and whole space we perform a rigorous analysis of the quasineutral limit for a hydrodynamical model of a viscous plasma with capillarity tensor represented by the Navier Stokes Korteweg Poisson system. We shall provide a detailed mathematical description of the convergence process by analyzing the dispersion of the fast oscillating acoustic waves. However the standard acoustic wave analysis is not sufficient to control the high frequency oscillations in the electric field but it is necessary to estimate the dispersive properties induced by the capillarity terms. Therefore by using these additional estimates we will be able to control, via compensated compactness, the quadratic nonlinearity of the stiff electric force field. In conclusion, opposite to the zero capillarity case [11] where persistent space localized time high frequency oscillations need to be taken into account, we show that as $\lambda \rightarrow 0$, the density fluctuation $\rho^\lambda - 1$ converges strongly to zero and the fluids behaves according to an incompressible dynamics.

Key words and phrases: compressible and incompressible Navier Stokes equation, Korteweg type fluids, energy estimates, dispersive equations and estimates, acoustic equation.

1991 Mathematics Subject Classification. Primary 35L65; Secondary 35L40, 76R50.

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1 Introduction and plan of the paper

1.1 Introduction

In the last years hydrodynamical models have been widely used to describe physical phenomena in plasma physics. In the particular case where the viscous stress tensors are taken into consideration the most simple model is provided by the coupling of the compressible Navier Stokes equations with the Poisson equation. In this case, in dimensionless units, the coupling can be expressed in terms of a constant λ which represents the scaled Debye length, a characteristic physical parameter for plasmas related to the phenomenon of the so called “Debye shielding”, [19]. Moreover willing take under consideration the surface tension effects it is necessary to add to the momentum equation a capillarity tensor, namely one has to consider Korteweg type model of capillarity. This type of models were first introduced by Korteweg [26], see also [30] and derived rigourosily by Dunn and Serrin [13] and are based on an extended version of thermodynamics which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density. Finally the model we will consider in this paper is given by the following Navier-Stokes -Poisson Korteweg system in $3 - D$, namely

$$\partial_t \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0, \quad (1)$$

$$\partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \nabla p(\rho^\lambda)^\gamma = \operatorname{div}(\mu \rho^\lambda D(u^\lambda) + K(\rho^\lambda)) + \rho^\lambda \nabla \Phi^\lambda, \quad (2)$$

$$\lambda^2 \Delta \Phi^\lambda = \rho^\lambda - 1, \quad (3)$$

where $p(\rho^\lambda)$ denotes the pressure term,

$$p(\rho^\lambda) = (\rho^\lambda)^\gamma, \quad \gamma \geq 3/2,$$

K the capillarity tensor which is given by

$$K_{ij}(\rho^\lambda) = \frac{\kappa}{2}(\Delta(\rho^\lambda)^2 - |\nabla \rho^\lambda|^2)\delta_{ij} - \kappa \partial_i \rho^\lambda \partial_j \rho^\lambda \quad (4)$$

and $D(u^\lambda)$ the strain tensor which has the form

$$D(u^\lambda)_{ij} = \frac{\partial_i u^\lambda_j + \partial_j u^\lambda_i}{2}. \quad (5)$$

Let $x \in \mathbb{R}^3$, $t \geq 0$, we denote by $\rho^\lambda(x, t)$ the *negative charge density*, by $m(x, t) = \rho^\lambda(x, t)u^\lambda(x, t)$ the *current density*, by $u^\lambda(x, t)$ the *velocity field*, by $\Phi^\lambda(x, t)$ the *electrostatic potential*, μ the *shear viscosity*. The parameter λ is the so called *Debye length* (up to a constant factor), κ is the capillarity coefficient. Moreover let us observe that

$$\operatorname{div} K(\rho^\lambda) = \rho^\lambda \nabla \Delta \rho^\lambda.$$

The purpose of this paper is to perform a rigorous limiting analysis when $\lambda \rightarrow 0$. The physical meaning of the Debye length λ is the distance over which the usual Coulomb field is killed off exponentially by the polarization of the plasma. In terms of physical variables the Debye length can be expressed as

$$\lambda = \lambda_D / L \quad \lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{e^2 n_0}}, \quad (6)$$

where L is the macroscopic length scale, ε_0 is the vacuum permittivity, k_B the Boltzmann constant, T the average plasma temperature, e the absolute electron charge and n_0 the average plasma density. In many cases the Debye length is very small compared to the macroscopic length $\lambda_D \ll L$ and so it makes sense to consider the quasineutral limit $\lambda \rightarrow 0$ of the system (1)-(3). In this situation the particle density is constrained to be close to the background density (equal to one in our case) of the oppositely charged particle. The limit $\lambda \rightarrow 0$ is called the quasineutral limit since the charge density almost vanishes identically. The velocity of the fluid then evolves according to an incompressible flow.

In the last years the quasineutral limit for hydrodynamical models of plasma or semiconductor devices has been widely studied by many authors, in the case of Euler Poisson system for instance by [7], [6], [29], [31] or the case of the Navier Stokes Poisson system by [35] who studied the quasineutral limit for the smooth solution with well-prepared initial data. Jiang and Wang [23] studied the combined quasineutral and inviscid limit of the compressible Navier- Stokes-Poisson system for weak solution and obtained the convergence of Navier- Stokes-Poisson system to the incompressible Euler equations with general initial data. Moreover in [23] the vanishing of viscosity coefficient was required in order to take the quasineutral limit and no convergence rate was derived there. The paper [24] studied the quasineutral limit of the isentropic Navier-Stokes-Poisson system both in the whole space and in the torus without restrictions on the viscous coefficients, with well prepared initial data.

The authors in [10] investigated the quasineutral limit of the isentropic Navier-Stokes-Poisson system in the whole space and obtained the convergence of weak solution of the Navier-Stokes-Poisson system to the weak solution of the incompressible Navier- Stokes equations by means of dispersive estimates of Strichartz's type under the assumption that the Mach number is related to the Debye length. A more general analysis in the context of weak solutions and in framework of general initial data was performed by the authors in [11] where all the regularity and smallness assumptions of the previous paper were removed. They were able to provide a detailed mathematical description of the convergence process by using microlocal defect measures and by developing an explicit correctors analysis.

Finally, in the contest of combined quasineutral and relaxation time limit

we have the papers by Gasser and Marcati in [15, 16, 17]. Other similar limits have been investigated in [5], [8], [9].

As far it concerns the compressible Navier Stokes Poisson Korteweg system we refer to [4] for the quasineutral limit in a periodic domain, where the electrons are assumed to be thermalized and to follow a non dimensional Maxwell-Boltzmann distribution ($\rho^\lambda = e^{\Phi^\lambda}$).

In all of these papers but [11], the assumptions are designed to kill the presence of high frequency oscillations of the electric force fields. In this paper we are interested to understand the limiting behaviour in the same general situation of [11], when a Korteweg tensor is added.

In this paper we perform the zero Debye limit for the Navier Stokes Poisson Korteweg system in the whole space and in the framework of weak solutions and large non smooth initial data. We do not make any particular assumption and our methods will control the quadratic stiff term of the electric fields by a better understanding of the role of the various scales in the oscillating wave packets.

A common feature for the limit analysis in the case of ill prepared initial data is the high plasma oscillations, namely the presence of high frequency time oscillations of the acoustic waves, moreover what actually makes the limiting behavior analysis very hard is the presence of very stiff terms due to the electric field, whose oscillations cannot be controlled only by the dispersion of the acoustic waves, as pointed out in [11]. In the case of fluids of Korteweg type an additional difficulty is represented by the loss of information on the gradient of the velocity when vacuum appears and the presence of these phenomena causes the lost of compactness for the momentum term. So it is particularly important to understand the different behaviors of the various vector fields acting in our system, what and which are the relationship between high frequency interacting waves, dispersive behavior and the different roles of time and space oscillations. There are distinct dispersive behavior acting on distinct scales and one has to analyze in detail their behavior.

The classical acoustic wave analysis is able control how the velocity field disperses and oscillates and in detail it follows by analyzing the dispersion of the acoustic equation related to the plasma fluctuation. We get that the dispersive behavior dominates on the high frequency time oscillations and usual estimates of Strichartz type are sufficient to pass into the limit of the convective term, but it is not able to control the electric fields time high frequency oscillations. The quadratic terms due to the electric force field may not be analyzed in the same way since the dispersion may no longer dominates the time high frequency wave packets and we have to take care of the self-interacting waves. The capillarity term induces additional dispersive effects on a different scale than the usual acoustic waves and by using non standard Strichartz estimates for the beam equations we can control the electric field. Intuitively while the acoustic waves scale like the standard

D'Alambert equation, the capillarity tensor induces linear dispersive waves which scale like the Klein-Gordon or the Schrödinger equation. The limit behavior of the quadratic nonlinearity of the electric force field is then deduced by a Compensated Compactness argument. The structure of this paper, as well as the main ingredients of our approach to this limiting process, can be summarized as follows. In Section 2 we set up our problem and state the main result. In Section 3 we collect the main mathematical tools needed in the paper, including notations and dispersive estimates. The Section 4 is devoted to obtain a priori estimates independent of λ , namely standard energy bounds. Section 5 concerns the convergence of the density. Section 6 and 7 are devoted to the convergence of the momentum and the electric field respectively. In that sections a careful analysis of the dispersion of the acoustic waves related to the plasma fluctuation will be performed. Finally, in Section 8 we conclude with the proof of our Main Theorem 1.

2 Statement of the problem and Main Result

Before performing our limiting analysis, we recast our problem in a more precise way and we recall some results concerning the existence of weak solutions for the Navier Stokes Poisson Korteweg system. The system under consideration in this paper is given by the following equations,

$$\begin{aligned}\partial_t \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) &= 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \nabla(\rho^\lambda)^\gamma &= \operatorname{div}(\mu \rho^\lambda D(u^\lambda)) + \rho^\lambda \nabla \Phi^\lambda + \kappa \rho^\lambda \nabla \Delta \rho^\lambda \\ \lambda^2 \Delta \Phi^\lambda &= \rho^\lambda - 1.\end{aligned}\tag{7}$$

From now on we set $\mu = \kappa = 1$ and we denote by π^λ the renormalized pressure,

$$\pi^\lambda = \frac{(\rho^\lambda)^\gamma - 1 - \gamma(\rho^\lambda - 1)}{(\gamma - 1)}.$$

Moreover we assume the initial data satisfy,

$$\begin{aligned}\rho_{t=0}^\lambda &= \rho_0^\lambda \geq 0, \quad \Phi|_{t=0} = \Phi_0^\lambda, \\ \rho^\lambda u^\lambda|_{t=0} &= m_0^\lambda, \quad m_0^\lambda = 0 \text{ on } \{x \in \mathbb{R}^3 \mid \rho_0^\lambda(x) = 0\}, \\ \mathcal{E}_0 &= \int_{\mathbb{R}^3} \left(\pi^\lambda|_{t=0} + \frac{|\nabla \rho_0^\lambda|^2}{2} + \frac{|m_0^\lambda|^2}{2\rho_0^\lambda} + \lambda^2 |\nabla \Phi_0^\lambda|^2 \right) dx < +\infty. \quad (\text{ID}) \\ \int_{\mathbb{R}^3} |\nabla \sqrt{\rho_0^\lambda}|^2 dx &< +\infty.\end{aligned}$$

Remark 2.1. By (ID) we get m_0^λ is bounded in $H^{-1}(\mathbb{R}^3)$. In fact we can rewrite m_0^λ in the following way

$$m_0^\lambda = \frac{m_0^\lambda}{\sqrt{\rho_0^\lambda}} \sqrt{\rho_0^\lambda} \chi_{|\rho_0^\lambda - 1| \leq 1/2} + \frac{m_0^\lambda}{\sqrt{\rho_0^\lambda}} \frac{\sqrt{\rho_0^\lambda}}{\sqrt{|\rho_0^\lambda - 1|}} \sqrt{|\rho_0^\lambda - 1|} \chi_{|\rho_0^\lambda - 1| > 1/2}$$

then, m_0^λ is bounded in $L^2(\mathbb{R}^3) + L^{2k/(k+1)}(\mathbb{R}^3)$ and hence in $H^{-1}(\mathbb{R}^3)$.

The existence of global weak solutions “à la Leray” for fixed $\lambda > 0$ for the system (7) deserves some comments. One of the main difficulty in the proof of existence of weak solutions for the Navier Stokes equations is the strong compactness of the density in some L^p space in order to pass to the limit in the pressure term. In the case of a Navier Stokes Korteweg fluid for the density are available bounds in $L^\infty((0, T); \dot{H}^1(\mathbb{R}^3))$, hence one can handle easily the convergence of the pressure term. However one is unable to pass to the limit in the quadratic terms of the type $\nabla \rho^\lambda \otimes \nabla \rho^\lambda$ appearing in the capillarity tensor and there is also a loss of information for the gradient of the velocity near the vacuum. The existence of strong solutions for the Navier Stokes Korteweg equations has been obtained by Hattori and Li in [21], [22], while the existence of weak solutions has been obtained by Bresch, Desjardins, Lin in [3] where they use some special test functions depending on ρ in order to deal with the vacuum problem. Their result can be easily adapted in order to prove the existence of weak solutions for the system (7). We summarize this existence result for the system (7) in the following theorem, see [4].

Theorem 2.2. *Assume (ID), and let $\gamma > 3/2$, then there exists a global weak solution $(\rho^\lambda, u^\lambda, \Phi^\lambda)$ to (7) such that $\rho^\lambda - 1 \in L^\infty((0, T); L_2^\gamma(\mathbb{R}^3))$, $\rho^\lambda \in L^2((0, T); \dot{H}^2(\mathbb{R}^3))$, $\nabla \rho^\lambda$ and $\nabla \sqrt{\rho^\lambda} \in L^\infty((0, T); L^2(\mathbb{R}^3))$, $\sqrt{\rho^\lambda} u^\lambda \in L^2((0, T); L^2(\mathbb{R}^3))$, $\sqrt{\rho^\lambda} D(u^\lambda) \in L^2((0, T); L^2(\mathbb{R}^3))$. Furthermore*

- *The following energy inequality holds for almost every $t \geq 0$,*

$$\mathcal{E}(t) + \int_0^t \int_{\mathbb{R}^3} \left(\mu |\sqrt{\rho^\lambda} D u^\lambda|^2 \right) dx ds \leq \mathcal{E}_0. \quad (8)$$

where

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \left(\rho^\lambda \frac{|u^\lambda|^2}{2} + \pi^\lambda + \frac{|\nabla \rho^\lambda|^2}{2} + \lambda^2 |\nabla \Phi^\lambda|^2 \right) dx.$$

- *The continuity equation (7)₁ is satisfied in the sense of distribution.*
- *For all $\varphi \in \mathcal{D}'((0, T) \times \mathbb{R}^3)$, one has*

$$\begin{aligned} & \int_{\mathbb{R}^3} m_0 \rho_0^\lambda u_0^\lambda + \int_0^T \int_{\mathbb{R}^3} \left((\rho^\lambda)^2 u^\lambda \cdot \partial_t \varphi + \rho^\lambda u^\lambda \otimes u^\lambda : D(\varphi) - (\rho^\lambda)^2 u^\lambda \cdot \varphi \operatorname{div} u^\lambda \right. \\ & \quad + (\rho^\lambda)^\gamma \operatorname{div} \varphi - (\rho^\lambda)^2 \nabla \Phi^\lambda \cdot \varphi - 2\rho^\lambda D(u^\lambda) : \rho^\lambda D(u^\lambda) \\ & \quad \left. - \rho^\lambda D(u^\lambda) : \varphi \otimes \nabla \rho^\lambda - (\rho^\lambda)^2 \Delta \rho^\lambda \operatorname{div} \varphi - 2\rho^\lambda (\varphi \cdot \nabla \rho) \Delta \rho^\lambda \right) dx dt = 0. \end{aligned}$$

Having collected all the preliminary material we are now ready to state our main result.

Main Theorem 1. *Let $(\rho^\lambda, u^\lambda, \Phi^\lambda)$ be a sequence of weak solutions in \mathbb{R}^3 of the system (7), assume that the initial data satisfy (ID). Then*

(i) $\rho^\lambda \longrightarrow 1$ weakly in $L^\infty([0, T]; L_2^k(\mathbb{R}^3))$ and strongly in $L^{2/s}((0, T); H_{loc}^{1+s}(\mathbb{R}^3)) \cap C((0, T); H_{loc}^s(\mathbb{R}^3))$, $0 < s < 1$.

(ii) The gradient component $\mathbf{H}^\perp(\rho^\lambda u^\lambda)$ of the momentum satisfies

$$\mathbf{H}^\perp(\rho^\lambda u^\lambda) \longrightarrow 0 \quad \text{strongly in } L^q([0, T]; L^p(\mathbb{R}^3)),$$

$$\text{where } p = \frac{4(s_0 + 3)}{4s_0 + 9}, \quad q = \frac{4(s_0 + 3)}{2s_0 + 5}, \quad \text{for any } s_0 \geq 3/2.$$

(iii) The divergence free component $\mathbf{H}(\rho^\lambda u^\lambda)$ of the momentum satisfies

$$\mathbf{H}(\rho^\lambda u^\lambda) \longrightarrow \mathbf{H}u = u \quad \text{strongly in } L^2([0, T]; L_{loc}^p(\mathbb{R}^3)), \quad 1 \leq p \leq 3/2.$$

(iv) $\rho^\lambda u^\lambda \longrightarrow u$ a.e.

(v) $u = \mathbf{H}u$ satisfies the following equation

$$\mathbf{H}(\partial_t u - \Delta u + (u \cdot \nabla)u) = 0, \tag{9}$$

$$\text{in } \mathcal{D}'([0, T] \times \mathbb{R}^3).$$

The remaining part of this paper is devoted to the proof of the Main Theorem 1.

3 Notations and Mathematical tools

For convenience of the reader we establish some notations and recall some basic facts that will be used in the sequel.

3.1 Notations

Given real valued functions F, G , we write $F \lesssim G$ if there exists $c \in \mathbb{R}$ such that $F \leq c G$.

We denote by

- a) $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+)$ the space of test function $C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, by $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$ the space of Schwartz distributions and $\langle \cdot, \cdot \rangle$ the duality bracket between \mathcal{D}' and \mathcal{D} .
- b) $W^{k,p}(\mathbb{R}^d) = (I - \Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ the nonhomogeneous Sobolev spaces, for any $1 \leq p \leq \infty$ and $k \in \mathbb{R}$. $\dot{W}^{k,p}(\mathbb{R}^d) = (-\Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $\dot{H}^k(\mathbb{R}^d) = \dot{W}^{k,2}(\mathbb{R}^d)$ denote the homogeneous Sobolev spaces. The notations $L_t^p L_x^q$, $L_t^p W_x^{k,q}$, $C_t W_x^{k,q}$ will abbreviate respectively the spaces $L^p([0, T]; L^q(\mathbb{R}^d))$, $L^p([0, T]; W^{k,q}(\mathbb{R}^d))$ and $C([0, T]; W^{k,q}(\mathbb{R}^d))$.

c) $L_2^p(\mathbb{R}^d)$ the Orlicz space defined as follows

$$L_2^p(\mathbb{R}^d) = \{f \in L_{loc}^1(\mathbb{R}^d) \mid |f|\chi_{|f| \leq \frac{1}{2}} \in L^2(\mathbb{R}^d), |f|\chi_{|f| > \frac{1}{2}} \in L^p(\mathbb{R}^d)\}, \quad (10)$$

see [1] for more details.

f) \mathbf{H} and \mathbf{H}^\perp the Helmotz Leray projectors, \mathbf{H}^\perp on the space of gradients vector fields and \mathbf{H} on the space of divergence - free vector fields. Namely

$$\mathbf{H}^\perp = \nabla \Delta^{-1} \operatorname{div}, \quad \mathbf{H} = \mathbf{I} - \mathbf{H}^\perp. \quad (11)$$

It is well known that \mathbf{H}^\perp and \mathbf{H} can be expressed in terms of Riesz multipliers, therefore they are bounded linear operators on every $W_x^{k,p}$ ($1 < p < \infty$) space (see [33]).

3.2 Mathematical tools

3.2.1 Compactness theorems

In the paper we use also the following compactness lemmas. The first one is the so called Lions-Aubin Lemma, (see [2], [32]).

Theorem 3.1. *Let be X, B, Y Banach spaces such that X is included in B with compact imbedding and $B \subset Y$ and let be u_n a bounded sequence in $L^p([0, T]; X)$, such that $\partial u_n / \partial t$ are bounded in $L^p([0, T]; Y)$ for $1 \leq p < \infty$. Then, u_n is relatively compact in $L^p([0, T]; B)$.*

We will also use the following generalization of the div-curl lemma (see Lemma 1.1. in [14])

Lemma 3.2. *Assume that $\{u_n(\cdot, t)\}$ and $\{v_n(\cdot, t)\}$ are vector fields in \mathbb{R}^d for $0 \leq t \leq T$ such that*

L1) $u_n \rightarrow u$ and $v_n \rightarrow v$ weak- in $L^\infty([0, T]; L_{loc}^2(\mathbb{R}^d))$ and strongly in $C([0, T]; H_{loc}^{-1}(\mathbb{R}^d))$;*

L2) $\{\operatorname{div} u_n\}$ is precompact in $C([0, T]; H_{loc}^{-1}(\mathbb{R}^d))$;

L3) $\{\operatorname{curl} v_n\}$ is precompact in $C([0, T]; H_{loc}^{-1}(\mathbb{R}^d))$.

Then,

$$u_n \cdot v_n \rightarrow u \cdot v \quad \text{in } \mathcal{D}'([0, T] \times \mathbb{R}^d).$$

3.2.2 Strichartz estimates for dispersive equations

As explained in the Introduction in the sequel we need dispersive estimates for equations describing the acoustic waves. So it is worthwhile to recall the basic facts for this equations. Following Keel and Tao [25], we start with a very general abstract setting.

Let (X, dx) be measure space and H a Hilbert space and for all $t \in \mathbb{R}$, $U(t) : H \rightarrow L^2(X)$ an operator that fulfills the following inequality

$$\|U(t)f\|_{L_x^2} \lesssim \|f\|_H \quad (12)$$

and for some $\delta > 0$ one of the following “dispersive” estimate.

For all $t \neq s$ and all $g \in L^1(X)$

$$\|U(s)(U(t))^*g\|_{L^\infty} \lesssim |t-s|^{-\delta}\|g\|_{L^1}. \quad (13)$$

For all t, s and all $g \in L^1(X)$

$$\|U(s)(U(t))^*g\|_{L^\infty} \lesssim (1+|t-s|)^{-\delta}\|g\|_{L^1}. \quad (14)$$

Definition 3.3. We say that the exponent pair (q, r) is δ -admissible if $q, r \leq 2$, $(q, r, \delta) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\delta}{r} \leq \frac{\delta}{2}. \quad (15)$$

If equality (15) hold we say that (q, r) is sharp δ -admissible.

We have then the following Strichartz type estimate (see [25]).

Theorem 3.4. *If $U(t)$ obeys (13) and (14), then the estimates*

$$\|U(t)f\|_{L_x^q L_t^r} \lesssim \|f\|_H \quad (16)$$

$$\left\| \int (U(s))^* F(s) ds \right\|_H \lesssim \|F\|_{L_x^{q'} L_t^{r'}} \quad (17)$$

$$\left\| \int U(t)(U(s))^* F(s) ds \right\|_{L_x^q L_t^r} \lesssim \|F\|_{L_x^{\tilde{q}'} L_t^{\tilde{r}'}} \quad (18)$$

hold for all sharp δ -admissible pairs (q, r) , (\tilde{q}, \tilde{r}) . Furthermore if the decay hypothesis is strengthened to (14), then (16), (17), (18) hold for all δ -admissible pairs (q, r) , (\tilde{q}, \tilde{r}) .

In the next to section we will apply the Theorem 3.4 to the dispersive equations used in the paper.

3.2.3 Strichartz estimate for Klein Gordon equation

We apply the previous Theorem 3.4 to the following Klein Gordon equation,

$$(-\partial_{tt} + \Delta - m^2) w(t, x) = F(t, x)$$

with Cauchy data

$$w(0, \cdot) = f, \quad \partial_t w(0, \cdot) = g,$$

where $m > 0$ is the mass and $0 < T < \infty$. It turns out that the Klein Gordon operator satisfies the decay estimate (13)-(14) with exponent $\delta = \frac{d-1}{2}$, so by applying the Theorem 3.4 with $d = 3$, $X = \mathbb{R}^3$ and $H = L^2(\mathbb{R}^3)$ we get that w satisfies the following Strichartz estimate, (see also [34])

$$\begin{aligned} \|w\|_{L_{t,x}^q} + \|\partial_t w\|_{L_t^q W_x^{-1,q}} + \|w\|_{C_t \dot{H}_x^{1/2}} + \|\partial_t w\|_{C_t \dot{H}_x^{-1/2}} \\ \lesssim \|f\|_{\dot{H}_x^{1/2}} + \|g\|_{\dot{H}_x^{-1/2}} + \|F\|_{L_{t,x}^p}, \end{aligned}$$

where (q, p) , are *admissible pairs*, namely they satisfy

$$\frac{4}{3} \leq p \leq \frac{10}{7} \quad \frac{10}{3} \leq q \leq 4.$$

By choosing $p = 4/3$ and $q = 4$ and by a standard application of Duhamel's principle it is straightforward to observe that for any $s \in \mathbb{R}$ the following Strichartz estimate holds,

$$\begin{aligned} \|w\|_{L_t^4 W_x^{s,4}} + \|\partial_t w\|_{L_t^4 W_x^{-1+s,4}} + \|w\|_{C_t \dot{H}_x^{1/2+s}} + \|w_t\|_{C_t \dot{H}_x^{-1/2+s}} \\ \lesssim \|f\|_{\dot{H}_x^{1/2+s}} + \|g\|_{\dot{H}_x^{-1/2+s}} + \|F\|_{L_t^1 \dot{H}_x^s}. \end{aligned} \tag{19}$$

3.2.4 Strichartz estimates for the Beam equations

The second dispersive equation we use is the so called Beam equation,

$$(-\partial_{tt} + \Delta^2 - m^2) w(t, x) = F(t, x)$$

with Cauchy data

$$w(0, \cdot) = f, \quad \partial_t w(0, \cdot) = g,$$

The Beam operator verifies the decay estimates (13)-(14) with and exponent $\delta = \frac{d}{4}$, see for example [20], [27]. Then by applying the Theorem 3.4 with $d = 3$, $X = \mathbb{R}^3$ and $H = L^2(\mathbb{R}^3)$ we get that w satisfies the following Strichartz estimate

$$\|w\|_{L_{t,x}^q} + \|w\|_{C_t L_x^2} + \|\partial_t w\|_{C_t \dot{H}_x^{-2}} \leq \|w_0\|_{\dot{H}^2} + \|w_1\|_{L_x^2} + \|F\|_{L_{t,x}^q}.$$

where q is *admissible*, namely it satisfy

$$q \geq 2 + \frac{8}{3}$$

As before, by a standard application of Duhamel's principle it is straightforward to see that for any $s \in \mathbb{R}$ the following estimate is also true

$$\|w\|_{W_{t,x}^{s,q}} + \|w\|_{C_t \dot{H}_x^s} + \|\partial_t w\|_{C_t \dot{H}_x^{s-2}} \leq \|w_0\|_{\dot{H}^{2+s}} + \|w_1\|_{\dot{H}_x^s} + \|F\|_{L_t^1 \dot{H}_x^s}. \quad (20)$$

4 Uniform estimates

In this section we wish to establish all the basic a priori estimates, independent on λ , for the solutions of the system (7). First of all we remind that from the Theorem 2.2 we have that the solutions of (7) satisfy the following uniform energy estimate

$$\mathcal{E}(t) + \int_0^t \int_{\mathbb{R}^3} \left(\mu |\sqrt{\rho^\lambda} D u^\lambda|^2 \right) dx ds \leq \mathcal{E}_0 \quad (21)$$

where

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \left(\rho^\lambda \frac{|u^\lambda|^2}{2} + \pi^\lambda + \frac{|\nabla \rho^\lambda|^2}{2} + \lambda^2 |\nabla \Phi^\lambda|^2 \right) dx.$$

Beside the standard estimate (21) it is possible to recover some further bounds on the second derivative of ρ^λ and on $\nabla \sqrt{\rho^\lambda}$. First, we need to prove the following lemma.

Lemma 4.1. *Assume that $(\rho^\lambda, u^\lambda, \Phi^\lambda)$ is a global weak solution of (7) and that (ID) hold, then the following identity holds,*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^\lambda |\nabla \log \rho^\lambda|^2 dx + \int_{\mathbb{R}^3} \nabla \operatorname{div} u^\lambda \cdot \nabla \rho^\lambda dx \\ + \int_{\mathbb{R}^3} \rho^\lambda D(u^\lambda) : \nabla \log \rho^\lambda \otimes \nabla \log \rho^\lambda = 0. \end{aligned} \quad (22)$$

Proof. The identity (22) follows easily by dividing first the continuity equation (7)₁ by ρ^λ , then by differentiating it with respect to space, and finally, by multiplying by $\rho^\lambda \partial_i \log \rho^\lambda$ and integrating by parts. \square

Using the identity (22) we are able to prove the following uniform estimate,

Proposition 4.2. *Assume that $(\rho^\lambda, u^\lambda, \Phi^\lambda)$ is a global weak solution of (7) and that (ID) hold, then the solutions of the system (7) satisfy the following inequality,*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left(\rho^\lambda |u^\lambda \nabla \log \rho^\lambda|^2 + |\nabla \rho^\lambda|^2 + \lambda |\nabla \Phi^\lambda|^2 \right) dx \\ + 4 \int_{\mathbb{R}^3} \left(p'(\rho^\lambda) |\nabla \sqrt{\rho^\lambda}|^2 + |\nabla \nabla \rho^\lambda|^2 + \frac{(\rho^\lambda - 1)^2}{\lambda^2} \right) dx \leq \mathcal{E}_0. \end{aligned} \quad (23)$$

Proof. By multiplying the momentum equation (7)₂ by $\nabla \rho^\lambda / \rho^\lambda$ and by integrating by parts we get

$$\begin{aligned} & \int_{\mathbb{R}^3} 4p'(\rho^\lambda) |\nabla \sqrt{\rho^\lambda}|^2 dx + \int_{\mathbb{R}^3} |\nabla \nabla \rho^\lambda|^2 dx \\ & + \int_{\mathbb{R}^3} \partial_t u^\lambda \nabla \rho^\lambda dx + \int_{\mathbb{R}^3} u^\lambda \nabla u^\lambda \nabla \rho^\lambda dx + \int_{\mathbb{R}^3} \nabla u^\lambda : \nabla \nabla \rho^\lambda dx \\ & - \int_{\mathbb{R}^3} \rho^\lambda \nabla u^\lambda : \nabla \log \rho^\lambda \otimes \nabla \log \rho^\lambda dx = \int_{\mathbb{R}^3} \nabla \Phi^\lambda \nabla \rho^\lambda dx. \end{aligned} \quad (24)$$

Now by using the Poisson equation (7)₃ we can rewrite the integral in the right hand side of (24) as follows,

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla \Phi^\lambda \nabla \rho^\lambda dx &= \int_{\mathbb{R}^3} \nabla \Phi^\lambda \nabla (\rho^\lambda - 1) dx \\ &= - \int_{\mathbb{R}^3} \Delta \Phi^\lambda (\rho^\lambda - 1) dx = - \int_{\mathbb{R}^3} \frac{(\rho^\lambda - 1)^2}{\lambda^2} dx \end{aligned} \quad (25)$$

Now by combining together (24) with (25) the identity (22) and the energy estimate (21) we end up with (23). \square

4.1 Consequences of the uniform estimate

We collect here all the a priori bounds provided by the energy inequality (21) and from the uniform estimate (23). From (21) we get that there exists $c > 0$ depending only from \mathcal{E}_0 , such that

$$\|\sqrt{\rho^\lambda} u^\lambda\|_{L_t^\infty L_x^2} \leq c, \quad (26)$$

$$\|\sqrt{\rho^\lambda} D(u^\lambda)\|_{L_t^2 L_x^2} \leq c, \quad (27)$$

$$\|\nabla \rho^\lambda\|_{L_t^\infty L_x^2} \leq c, \quad (28)$$

$$\|\lambda \nabla \Phi^\lambda\|_{L_t^\infty L_x^2} \leq c. \quad (29)$$

Moreover, since $\pi^\lambda \in L^\infty([0, T]; L^1(\mathbb{R}^3))$ it is straightforward to deduce

$$\rho^\lambda - 1 \text{ is bounded in } L^\infty([0, T]; L_2^k(\mathbb{R}^3)), \text{ where } k = \min(\gamma, 2). \quad (30)$$

The additional estimate (23) provides more regularity on ρ^λ , in fact we have that

$$\|\rho^\lambda\|_{L_t^2 \dot{H}_x^2} \leq c. \quad (31)$$

The uniform $L^\infty([0, T]; L^2(\mathbb{R}^3))$ bound on $\sqrt{\rho^\lambda} \nabla \log \rho^\lambda$ yields to

$$\|\nabla \sqrt{\rho^\lambda}\|_{L_t^\infty L_x^2} \leq c. \quad (32)$$

Finally, from (23) it follows

$$\|\rho^\lambda - 1\|_{L_t^2 L_x^2} \leq c\lambda. \quad (33)$$

5 Strong convergence of ρ^λ and $\sqrt{\rho^\lambda}$

Here by using the bounds obtained in Section 4 we are able to prove some results concerning the convergence of ρ^λ and $\sqrt{\rho^\lambda}$.

5.1 Convergence of ρ^λ

A straightforward consequence of (33) is that

$$\rho^\lambda \longrightarrow 1 \quad \text{strongly in } L_t^2 L_x^2. \quad (34)$$

Now by using together (28), (31) and the fact that

$$\partial_t \rho^\lambda = -\operatorname{div}(\rho^\lambda u^\lambda) \quad \text{is bounded in } L_t^2 H_x^{-1},$$

we can apply the Lions-Aubin Lemma 3.1 to conclude that

$$\begin{aligned} \rho^\lambda &\longrightarrow 1 \quad \text{strongly in} \\ &L^{2/s}(0, T; H^{1+s}(K) \cap C(0, T; H^s(K)), \quad 0 < s < 1 \\ \text{and} \\ \rho^\lambda &\longrightarrow 1 \quad \text{uniformly in } [0, T] \times K, \end{aligned} \quad (35)$$

where $K \subset \mathbb{R}^3$ is a compact set. Finally, by combining together (28), (31) and (35) we get that

$$\nabla \rho^\lambda \longrightarrow 0 \quad \text{strongly in } L_t^2 L_x^2. \quad (36)$$

5.2 Convergence of $\sqrt{\rho^\lambda}$

From (32) and the mass conservation we know that $\sqrt{\rho^\lambda}$ is bounded in $L_t^\infty H_x^1$, next we observe that

$$\partial_t \sqrt{\rho^\lambda} = -\frac{1}{2\sqrt{\rho^\lambda}} \operatorname{div}(\rho^\lambda u^\lambda) = \frac{1}{2} \sqrt{\rho^\lambda} \operatorname{div} u^\lambda - \operatorname{div}(\sqrt{\rho^\lambda} u^\lambda)$$

so by using (26) and (27) we have that $\partial_t \sqrt{\rho^\lambda}$ is bounded in $L_t^2 H_x^{-1}$. By applying the Lions-Aubin Lemma 3.1 we get that

$$\sqrt{\rho^\lambda} \longrightarrow 1 \quad \text{strongly in } L^2(0, T; L_{loc}^2(\mathbb{R}^3)) \quad (37)$$

Next, by using (34) and (36) we have that

$$\nabla \sqrt{\rho^\lambda} \longrightarrow 0 \quad \text{strongly in } L^2(0, T; L_{loc}^2(\mathbb{R}^3)). \quad (38)$$

6 Convergence and Compactness of $\rho^\lambda u^\lambda$

The next step in our limit analysis is to get enough information in order to pass into the limit in the convective term $\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)$. Unfortunately the estimates of the Section 4 are not enough to handle this nonlinear term. In fact from the bound (26) we get only the weak convergence of $\sqrt{\rho^\lambda} u^\lambda$ and a new difficulty takes place concerning the loss of information on the gradient of u^λ (see the estimate (27)) when vacuum appears. So it becomes involved to pass to the limit in the term $\rho^\lambda u^\lambda \otimes u^\lambda$. In order to deal with this loss of information the goal of this section is to get some compactness for the momentum term $\rho^\lambda u^\lambda$. First of all, by combining together (26) and (32) we have that

$$\rho^\lambda u^\lambda \quad \text{is uniformly bounded in } L_t^\infty L_x^p, \quad 1 \leq p \leq 3/2. \quad (39)$$

On the other hand from (26), (27), (32) we get

$$\nabla(\rho^\lambda u^\lambda) \quad \text{is uniformly bounded in } L_t^2 L_x^1, \quad (40)$$

hence, by using simultaneously (39) and (40) we have

$$\rho^\lambda u^\lambda \quad \text{is uniformly bounded in } L_t^2 W_x^{1,1}. \quad (41)$$

The previous bound entails only the weak convergence of $\rho^\lambda u^\lambda$ in $L_t^2 W_x^{1,1}$. In order to study the strong convergence we decompose the momentum term in its soleinoidal and gradient part, namely

$$\rho^\lambda u^\lambda = \mathbf{H}(\rho^\lambda u^\lambda) + \mathbf{H}^\perp(\rho^\lambda u^\lambda)$$

and we analyze separately the convergence of these two terms.

6.1 Compactness of the soleinoidal part $\mathbf{H}(\rho^\lambda u^\lambda)$

In order to get the compactness of $\mathbf{H}(\rho^\lambda u^\lambda)$, by using the Lions-Aubin Lemma 3.1 and (41), we need to show that $\partial_t \mathbf{H}(\rho^\lambda u^\lambda)$ is bounded in $L_t^2 W_x^{-k,p}$ for some $k > 0$ and $p \geq 1$. From the uniform bounds of Section 4 we have

$$\operatorname{div}(\sqrt{\rho^\lambda} u^\lambda \otimes \sqrt{\rho^\lambda} u^\lambda), \quad \nabla p(\rho^\lambda) \in L_t^\infty W_x^{-1,1}, \quad (42)$$

$$(\rho^\lambda - 1) \nabla \Phi^\lambda = \operatorname{div}(\lambda \nabla \Phi^\lambda \otimes \lambda \nabla \Phi^\lambda) + \frac{\lambda^2}{2} \nabla |\nabla \Phi^\lambda|^2 \in L_t^\infty W_x^{-1,1}, \quad (43)$$

$$(\rho^\lambda - 1) \nabla \Delta \rho^\lambda \in L_t^\infty W_x^{-1,2}. \quad (44)$$

By applying the Helmotz-Leray projector \mathbf{H} to the momentum equation we are able to conclude

$$\partial_t \mathbf{H}(\rho^\lambda u^\lambda) \quad \text{is bounded in } L^2([0, T]; W^{-2,4/3}(\mathbb{R}^3)). \quad (45)$$

Finally, (45) and the Lemma 3.1 yields

$$\mathbf{H}(\rho^\lambda u^\lambda) \text{ is compact in } L^2([0, T]; L_{loc}^p(\mathbb{R}^3)), \quad 1 \leq p \leq 3/2. \quad (46)$$

Next, since $\sqrt{\rho^\lambda} u^\lambda$ is uniformly bounded in $L_t^2 L_x^2$ we deduce that it converges weakly to some $\overline{m} \in L_t^2 L_x^2$. This fact together with (35) allows us to define a limit velocity u as follows

$$u^\lambda(t, x) = \frac{\sqrt{\rho^\lambda} u^\lambda}{\sqrt{\rho^\lambda}} \rightharpoonup \overline{m} = u(t, x) \quad \text{in } L_t^2 L_x^2. \quad (47)$$

and by combining (35), together with (27) and (49) we have for a.e $t \in [0, T]$

$$u^\lambda(t, \cdot) \longrightarrow \overline{m} \quad \text{strongly in } L_{loc}^2.$$

Hence by passing into the limit inside the conservation of mass equation (7)₁ we get

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^3). \quad (48)$$

By using together (35), (46), (48), it follows

$$\mathbf{H}(\rho^\lambda u^\lambda) \longrightarrow \mathbf{H}u = u \quad \text{strongly in } L^2([0, T]; L_{loc}^p(\mathbb{R}^3)), \quad 1 \leq p \leq 3/2. \quad (49)$$

6.2 Convergence of $\mathbf{H}^\perp(\rho^\lambda u^\lambda)$

Let us define the density fluctuation in the usual way

$$\sigma^\lambda = \frac{\rho^\lambda - 1}{\lambda} \quad (50)$$

then, by the identity

$$\mathbf{H}^\perp(\rho^\lambda u^\lambda) = -\lambda \nabla \Delta^{-1} \partial_t \sigma^\lambda, \quad (51)$$

we can deduce that the convergence of $\mathbf{H}^\perp(\rho^\lambda u^\lambda)$ is strictly related to the one of the density fluctuation. As mentioned in the introduction the weak convergence of the the gradient part of $\rho^\lambda u^\lambda$ is induced by the so called acoustic waves. In fact as we will see in this sections the density fluctuation exhibits very fast oscillating waves in time (the so called plasma oscillation). In order to control this high frequency waves we will recover the acoustic equation satisfied by σ^λ , we show that it enjoys various dispersive properties which will enable us to estimate the density fluctuation σ^λ uniformly with respect to λ .

Let us rewrite the system (7) in the following way

$$\partial_t \sigma^\lambda + \frac{1}{\lambda} \operatorname{div}(\rho^\lambda u^\lambda) = 0 \quad (52)$$

$$\begin{aligned} \partial_t(\rho^\lambda u^\lambda) + \frac{1}{\lambda} \nabla \sigma^\lambda &= \operatorname{div}(\rho^\lambda D(u^\lambda)) - \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) - \nabla p(\rho^\lambda) \\ &\quad + \frac{1}{\lambda} \nabla \sigma^\lambda + (\rho^\lambda - 1) \nabla \Phi^\lambda + \nabla \Phi^\lambda + \rho^\lambda \nabla \Delta \rho^\lambda, \end{aligned} \quad (53)$$

$$\lambda \Delta \Phi^\lambda = \sigma^\lambda. \quad (54)$$

Then, by differentiating with respect to time the equation (52), by taking the divergence of (53) and by using (54) we get that σ^λ satisfies the following equation

$$\begin{aligned} \lambda^2 \partial_{tt} \sigma^\lambda - \Delta \sigma^\lambda + \sigma^\lambda &= \lambda \operatorname{div} \operatorname{div} \left(\rho^\lambda D(u^\lambda) - \rho^\lambda u^\lambda \otimes u^\lambda \right) \\ &+ \lambda \operatorname{div} \left(-\nabla p(\rho^\lambda) + \frac{1}{\lambda} \nabla \sigma^\lambda + (\rho^\lambda - 1) \nabla \Phi^\lambda + \rho^\lambda \nabla \Delta \rho^\lambda \right). \end{aligned} \quad (55)$$

It turns out that (55) is a nonhomogeneous Klein Gordon equation with mass $1/\lambda$, in order to get uniform bounds on the fluctuation σ^λ we have to take into account the combined description of dispersion and high frequency time oscillations provided by the Strichartz estimates (19). In order to make the equation (55) more easier to handle, we rescale the time variable, the density fluctuation, the velocity and the electric potential in the following way

$$\tau = \frac{t}{\lambda}, \quad y = x \quad (56)$$

$$\begin{aligned} \tilde{u}(y, \tau) &= u^\lambda(y, \lambda\tau), \quad \tilde{\rho}(y, \tau) = \rho^\lambda(y, \lambda\tau) \\ \tilde{\sigma}(y, \tau) &= \sigma^\lambda(y, \lambda\tau), \quad \tilde{\Phi}(y, \tau) = \Phi^\lambda(y, \lambda\tau). \end{aligned} \quad (57)$$

As a consequence of this scaling the Klein Gordon equation (55) becomes,

$$\partial_{\tau\tau} \tilde{\sigma} - \Delta \tilde{\sigma} + \tilde{\sigma} = \tilde{F} \quad (58)$$

where

$$\begin{aligned} \tilde{F} &= -\lambda \operatorname{div} \left(\operatorname{div}(\tilde{\rho} D(\tilde{u})) - \operatorname{div}(\tilde{\rho} \tilde{u} \otimes \tilde{u}) - \nabla p(\tilde{\rho}) + (\tilde{\rho} - 1) \nabla \tilde{\Phi} \right) \\ &+ \lambda \operatorname{div} ((\tilde{\rho} - 1) \nabla \Delta \tilde{\rho}) + \lambda \operatorname{div} (\nabla \Delta \tilde{\rho} + \lambda^{-1} \nabla \tilde{\sigma}) \\ &= \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3. \end{aligned} \quad (59)$$

By using the uniform bounds of the Section 4, for any $s_0 \geq 3/2$ we have

$$\begin{aligned} \tilde{F}_1 &= \lambda \operatorname{div} \left(\operatorname{div}(\tilde{\rho} D(\tilde{u})) - \operatorname{div}(\tilde{\rho} \tilde{u} \otimes \tilde{u}) - \nabla p(\tilde{\rho}) \right. \\ &\quad \left. + \operatorname{div}(\lambda \nabla \tilde{\Phi} \otimes \lambda \nabla \tilde{\Phi}) + \frac{1}{2} \nabla |\lambda \nabla \tilde{\Phi}|^2 \right) \in L_t^\infty H_x^{-s_0-2}, \end{aligned} \quad (60)$$

$$\tilde{F}_2 = \lambda^2 \operatorname{div} \nabla (\tilde{\sigma} \Delta \rho^\lambda) + \lambda \operatorname{div} (\nabla \tilde{\rho} \Delta \tilde{\rho}) \in L_t^1 H_x^{-s_0-2} + L_t^2 H_x^{-s_0-1}, \quad (61)$$

$$\tilde{F}_3 = \lambda \operatorname{div} (\nabla \Delta \tilde{\rho}) + \operatorname{div} (\nabla \tilde{\sigma}) \in L_t^2 H_x^{-2}. \quad (62)$$

Then the following estimate on σ^λ holds.

Theorem 6.1. *Let us consider the solutions $(\rho^\lambda, u^\lambda, \Phi^\lambda)$ of the Cauchy problem for the system (7) with initial data satisfying (ID). Then, for any $s_0 \geq 3/2$, the following estimate holds*

$$\begin{aligned}
& \lambda^{-\frac{1}{4}} \|\sigma^\lambda\|_{L_t^4 W_x^{-s_0-2,4}} + \lambda^{\frac{3}{4}} \|\partial_t \sigma^\lambda\|_{L_t^4 W_x^{-s_0-3,4}} \\
& + \|\sigma^\lambda\|_{C_t H_x^{-3/2-s_0}} + \lambda \|\partial_t \sigma^\lambda\|_{C_t H_x^{-5/2-s_0}} \\
& \lesssim \|\sigma_0^\lambda\|_{H_x^{-1}} + \|m_0^\lambda\|_{H_x^{-1}} \\
& + T \|\operatorname{div}(\operatorname{div}(\rho^\lambda D(u^\lambda)) - \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) - \nabla p(\rho^\lambda) + (\rho^\lambda - 1) \nabla \Phi^\lambda)\|_{L_t^\infty H_x^{-s_0-2}} \\
& + \lambda \|\operatorname{div} \nabla(\sigma^\lambda \Delta \rho^\lambda)\|_{L_t^1 H_x^{-s_0-2}} + T \|\operatorname{div}(\nabla(\rho^\lambda - 1) \Delta \rho^\lambda)\|_{L_t^2 H_x^{-s_0-1}} \\
& + \sqrt{T} \|\operatorname{div}(\nabla \Delta \rho^\lambda + \lambda^{-1} \nabla(\rho^\lambda - 1))\|_{L_t^2 H_x^{-2}}.
\end{aligned} \tag{63}$$

Proof. By using the bounds (60)-(62) and in the same spirit of [11], we apply the Strichartz estimate (19) with $(y, \tau) \in \mathbb{R}^3 \times (0, T/\lambda)$ to the scaled Klein Gordon equation (58) and we get

$$\begin{aligned}
& \|\tilde{\sigma}\|_{L_\tau^4 W_x^{-s_0-2,4}} + \|\partial_\tau \tilde{\sigma}\|_{L_\tau^4 W_y^{-s_0-3,4}} \\
& + \|\tilde{\sigma}\|_{C_\tau H_x^{-3/2-s_0}} + \|\partial_\tau \tilde{\sigma}\|_{C_\tau H_x^{-5/2-s_0}} \\
& \lesssim \|\tilde{\sigma}_0\|_{H_y^{-3/2-s_0}} + \|\partial_\tau \tilde{\sigma}_0\|_{H_y^{-5/2-s_0}} \\
& + T \|\operatorname{div}(\operatorname{div}(\tilde{\rho} \tilde{u} \otimes \tilde{u}) + \nabla p(\tilde{\rho}) + \operatorname{div}(\tilde{\rho} D(\tilde{u})) + (\tilde{\rho} - 1) \nabla \tilde{\Phi})\|_{L_\tau^\infty H_y^{-s_0-2}} \\
& + \lambda^2 \|\operatorname{div} \nabla(\tilde{\sigma} \Delta \tilde{\rho})\|_{L_\tau^1 H_y^{-s_0-2}} + \sqrt{\lambda} \sqrt{T} \|\operatorname{div}(\nabla(\tilde{\rho} - 1) \Delta \tilde{\rho})\|_{L_\tau^2 H_y^{-s_0-1}} \\
& + \sqrt{\lambda} \sqrt{T} \|\operatorname{div}(\nabla \Delta \tilde{\rho} + \lambda^{-1} \nabla(\tilde{\rho} - 1))\|_{L_\tau^2 H_y^2}.
\end{aligned}$$

Finally, since

$$\|\tilde{\sigma}\|_{L_\tau^q W_y^{k,p}} = \lambda^{-\frac{1}{q}} \|\tilde{\sigma}\|_{L_t^q W_x^{k,p}}$$

by the Remark 2.1 and $\sigma_0 = \lambda \Delta \Phi_0 \in H_x^{-1}$ we end up with (64). \square

Going back to (51) and by using (64) we get

$$\mathbf{H}^\perp(\rho^\lambda u^\lambda) \longrightarrow 0 \quad \text{strongly in } L_t^4 W_x^{-s_0-2,4}, \text{ for any } s_0 \geq 3/2. \tag{65}$$

By using (40) and (66), by standard interpolation it follows

$$\mathbf{H}^\perp(\rho^\lambda u^\lambda) \longrightarrow 0 \quad \text{strongly in } L^q([0, T]; L^p(\mathbb{R}^3)), \tag{66}$$

where $p = \frac{4(s_0 + 3)}{4s_0 + 9}$, $q = \frac{4(s_0 + 3)}{2s_0 + 5}$, for any $s_0 \geq 3/2$.

7 Convergence of the electric field

This section is devoted to the study of the convergence of the electric field $E^\lambda = \nabla \Phi^\lambda$. By the a priori estimate (29) we know that λE^λ is bounded in $L_t^\infty L_x^2$ which does not give enough information to pass into the limit in the quadratic term $\rho^\lambda \nabla \Phi^\lambda \sim \operatorname{div}(\lambda E^\lambda \otimes \lambda E^\lambda) - 1/2 \nabla |\lambda E^\lambda|^2$, appearing in the righthand side of (7)₂. Hence, the problem now, is how to recover the weak continuity of this quadratic forms in L^2 . A way to recover some weak continuity for scalar product of L^2 sequences is given by a compensated compactness tool as the div-curl lemma. For this purpose we have to recover compactness in space and time. A key observation follows from the Poisson equation (7)₃ written in terms of electric field and density fluctuation

$$\lambda E^\lambda = \nabla \Delta^{-1} \sigma^\lambda, \quad (67)$$

where, by using (29) and (33) we have

$$\lambda E^\lambda \quad \text{is bounded in } L^2(0, T; H^1(\mathbb{R}^3)). \quad (68)$$

The previous bound gives us compactness in space but not in time. On the other hand the dispersion of the acoustic equation of Klein Gordon type does not gives us sufficient information. One way to overcome this further difficulty would be to exploit in a better way the dispersive behavior of all the terms appearing in the momentum equation. In the previous section we focused on the dispersion given by the combination of the electric field and the Poisson equation, now we are going to exploit the dispersive properties induced by the capillarity term $\rho^\lambda \nabla \Delta \rho^\lambda$. This will be done in the next section.

7.1 Beam equation for the density fluctuation

We rewrite the system (7) in the following way

$$\partial_t \sigma^\lambda + \frac{1}{\lambda} \operatorname{div}(\rho^\lambda u^\lambda) = 0 \quad (69)$$

$$\begin{aligned} \partial_t(\rho^\lambda u^\lambda) &= \operatorname{div}(\rho^\lambda D(u^\lambda)) - \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) - \nabla p(\rho^\lambda) \\ &\quad + (\rho^\lambda - 1) \nabla \Phi^\lambda + \nabla \Phi^\lambda + (\rho^\lambda - 1) \nabla \Delta \rho^\lambda + \nabla \Delta \rho^\lambda, \end{aligned} \quad (70)$$

$$\lambda \Delta \Phi^\lambda = \sigma^\lambda. \quad (71)$$

By differentiating (69) with respect to time and by taking the divergence of (70) we get

$$\begin{aligned} \partial_{tt} \sigma^\lambda + \Delta^2 \sigma^\lambda + \frac{1}{\lambda^2} \sigma^\lambda &= \frac{1}{\lambda} \operatorname{div} \left(\rho^\lambda D(u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \nabla p(\rho^\lambda) \right) \\ &\quad + \frac{1}{\lambda} \operatorname{div} \left((\rho^\lambda - 1) \nabla \Phi^\lambda + (\rho^\lambda - 1) \nabla \Delta \rho^\lambda \right). \end{aligned} \quad (72)$$

The equation (72) goes under the name of Beam equation. In order to handle it in a more easier way we rescale the time and space variables, the density fluctuation, the velocity and the electric potential in the following way

$$\tau = \frac{t}{\lambda}, \quad y = \frac{x}{\sqrt{\lambda}} \quad (73)$$

$$\begin{aligned} \tilde{u}(y, \tau) &= u^\lambda(\sqrt{\lambda}y, \lambda\tau), & \tilde{\rho}(y, \tau) &= \rho^\lambda(\sqrt{\lambda}y, \lambda\tau) \\ \tilde{\sigma}(y, \tau) &= \sigma^\lambda(\sqrt{\lambda}y, \lambda\tau), & \tilde{\Phi}(y, \tau) &= \Phi^\lambda(\sqrt{\lambda}y, \lambda\tau). \end{aligned} \quad (74)$$

Then the equation (72) becomes

$$\partial_{tt}\tilde{\sigma} + \Delta^2\tilde{\sigma} + \tilde{\sigma} = \tilde{F} \quad (75)$$

where

$$\begin{aligned} \tilde{F} &= \operatorname{div}(\operatorname{div}(\tilde{\rho}D(\tilde{u})) - \operatorname{div}(\tilde{\rho}\tilde{u} \otimes \tilde{u}) - \nabla p(\tilde{\rho})) \\ &\quad + \operatorname{div}\left((\tilde{\rho} - 1)\nabla\tilde{\Phi}\right) + \frac{1}{\lambda}\operatorname{div}\left((\tilde{\rho} - 1)\nabla\Delta\tilde{\rho}\right) \\ &= \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3. \end{aligned} \quad (76)$$

By taking into account the scaling (73) and (74) and the uniform bounds of Section 4 for any $s_0 \geq 3/2$ we have

$$\tilde{F}_1 = \operatorname{div}(\operatorname{div}(\tilde{\rho}D(\tilde{u})) - \operatorname{div}(\tilde{\rho}\tilde{u} \otimes \tilde{u}) - \nabla p(\tilde{\rho})) \in L_t^\infty H_x^{-s_0-2}, \quad (77)$$

$$\tilde{F}_2 = \operatorname{div}(\operatorname{div}(\sqrt{\lambda}\nabla\tilde{\Phi} \otimes \sqrt{\lambda}\nabla\tilde{\Phi}) + 1/2\nabla|\sqrt{\lambda}\nabla\tilde{\Phi}|^2) \in L_t^\infty H_x^{-s_0-2}, \quad (78)$$

$$\lambda\tilde{F}_3 = \operatorname{div}\nabla((\tilde{\rho} - 1)\Delta\tilde{\rho}) + \operatorname{div}(\nabla\tilde{\rho}\Delta\tilde{\rho}) \in L_t^1 H_x^{-s_0-2} + L_t^2 H_x^{-s_0-2} \quad (79)$$

By using the Strichartz estimates for the Beam equation we are able to prove the following theorem

Theorem 7.1. *Let us consider the solutions $(\rho^\lambda, u^\lambda, V^\lambda)$ of the Cauchy problem for the system (7) with initial data satisfying (ID). Then for any*

$s_0 \geq 3/2$, the following estimate holds

$$\begin{aligned}
& \lambda^{-\frac{1}{4}-\frac{5}{2q}} \|\sigma^\lambda\|_{L_t^q \dot{W}_x^{-s_0-2,q}} + \|\sigma^\lambda\|_{C(0,T;\dot{H}_x^{-s_0-2})} + \|\partial_t \sigma^\lambda\|_{C(0,T;\dot{H}_x^{-s_0-4})} \\
& \lesssim \lambda \|\sigma_0^\lambda\|_{H_x^{-1}} + \|m_0^\lambda\|_{H_x^{-1}} \\
& + T \|\operatorname{div}(\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) - \nabla p(\rho^\lambda) + \operatorname{div}(\rho^\lambda Du^\lambda))\|_{L_t^\infty \dot{H}_x^{-s_0-2}} \\
& + T \|\operatorname{div}(\operatorname{div}(\lambda \nabla \Phi^\lambda \otimes \nabla \Phi^\lambda) + \frac{1}{2} \nabla |\lambda \nabla \Phi^\lambda|^2)\|_{L_t^\infty \dot{H}_x^{-s_0-2}} \\
& + \|\lambda^{-1} \operatorname{div} \nabla ((\rho^\lambda - 1) \Delta \rho^\lambda)\|_{L_t^1 \dot{H}_x^{-s_0-2}} + \sqrt{T} \sqrt{\lambda} \|\operatorname{div} \nabla \tilde{\rho} \Delta \tilde{\rho}\|_{L_t^2 \dot{H}_x^{-2}}. \quad (80)
\end{aligned}$$

Proof. By using the bounds (77)-(79), as in the previous section, we apply the Strichartz estimate (20) with $(y, \tau) \in \mathbb{R}^3 \times (0, T/\lambda)$ to the scaled Beam equation (75) and we get that $\tilde{\sigma}$ verifies

$$\begin{aligned}
& \|\tilde{\sigma}\|_{L_\tau^q W_x^{-s_0-2,q}} + \|\tilde{\sigma}\|_{C(0,T;H_x^{-s_0-2})} + \|\partial_t \tilde{\sigma}\|_{C(0,T;H_x^{-s_0-4})} \\
& \lesssim \|\tilde{\sigma}_0\|_{H_y^{-s_0}} + \|\partial_\tau \tilde{\sigma}_0\|_{H_y^{-s_0-2}} \\
& + \frac{T}{\lambda} \|\operatorname{div}(\operatorname{div}(\tilde{\rho} \tilde{u} \otimes \tilde{u}) + \nabla \tilde{\pi} + \tilde{\rho} D(\tilde{u}))\|_{L_\tau^\infty H_y^{-s_0-2}} \\
& + \frac{T}{\lambda} \|\operatorname{div}(\operatorname{div}(\sqrt{\lambda} \nabla \tilde{\Phi} \otimes \sqrt{\lambda} \nabla \tilde{\Phi}) + \frac{1}{2} \nabla |\sqrt{\lambda} \nabla \tilde{\Phi}|^2)\|_{L_\tau^\infty H_y^{-s_0-2}} \\
& + \frac{1}{\sqrt{\lambda}} \|\operatorname{div} \nabla \left(\frac{(\tilde{\rho} - 1)}{\lambda^{1/4}} \frac{\Delta \tilde{\rho}}{\lambda^{1/4}} \right)\|_{L_\tau^1 H_y^{-s_0-2}} + \frac{\sqrt{T}}{\sqrt{\lambda} \lambda^{1/4}} \|\operatorname{div} \left(\frac{\nabla \tilde{\rho}}{\sqrt{\lambda}} \frac{\Delta \tilde{\rho}}{\lambda^{1/4}} \right)\|_{L_\tau^2 H_y^{-s_0-2}}
\end{aligned}$$

Finally, since

$$\|\tilde{\sigma}\|_{L_\tau^q W_y^{k,p}} = \lambda^{-\frac{1}{q} + \frac{k}{2} - \frac{3}{2p}} \|\sigma^\varepsilon\|_{L_t^p W_x^{k,q}}$$

and, as before, by using the Remark 2.1 and the fact that $\sigma_0 = \lambda \Delta \Phi_0 \in H_x^{-1}$ we end up with (80). \square

With the uniform estimate (80) we are able to prove the following compactness results concerning $\lambda \nabla \Phi^\lambda$.

Proposition 7.2. *Let $(\rho^\lambda, u^\lambda, \Phi^\lambda)$ be a sequence of solutions of the Navier Stokes Korteweg Poisson system (7) which satisfy (ID), then it holds*

$$\lambda \nabla \Phi^\lambda \otimes \lambda \nabla \Phi^\lambda \rightharpoonup 0 \quad \text{in } \mathcal{D}'([0, T] \times \mathbb{R}^3). \quad (81)$$

Proof. In order to prove (81) we apply the generalized Div-curl Lemma 3.2 to the sequences $u_n = v_n = \lambda \nabla \Phi^\lambda = E^\lambda$. So we have to check that the hypotheses L1 – L3 hold. By combining together (26), (67) and (80) we get that

$$\lambda E^\lambda \longrightarrow 0 \quad \text{weak-* in } L^\infty([0, T]; L_{loc}^2(\mathbb{R}^3)). \quad (82)$$

Then, we observe that the hypothesis $L3$ is automatically fulfilled since $\operatorname{curl} E^\lambda = 0$. In order to verify the hypothesis $L2$ we see that by the Poisson equation

$$\operatorname{div} E^\lambda = \lambda \nabla \Phi^\lambda = \sigma^\lambda.$$

By using (80) we have $\partial_t \sigma^\lambda \in C(0, T; H^{-s}(\mathbb{R}^3))$, for any $s > 1$, so σ^λ is bounded in $Lip(0, T; H^{-s}(\mathbb{R}^3))$, $s > 1$ which together with the energy bounds on σ^λ in $L^2(0, T; L^2(\mathbb{R}^3))$, yields to the precompactness of $\operatorname{div} E^\lambda$ in $C([0, T]; H_{loc}^{-1}(\mathbb{R}^3))$. In a similar way we fulfill the hypothesis $L1$, by combining $\lambda E^\lambda = \Delta^{-1/2} \sigma^\lambda$ and the bounds (80). Since $L1$ - $L3$ holds we can conclude by using the Lemma 3.2 that

$$\lambda \nabla \Phi^\lambda \otimes \lambda \nabla \Phi^\lambda \rightharpoonup 0 \quad \text{in } \mathcal{D}'([0, T] \times \mathbb{R}^3).$$

□

8 Proof of the Main Theorem 1

- (i) It follows from (30) and (35).
- (ii) It follows from (66).
- (iii) It follows from (49).
- (iv) It follows from (ii) and (iii).
- (v) First of all we apply the Leray projector \mathbf{H} to the momentum equation of the system (7), then we have

$$\begin{aligned} & \partial_t \mathbf{H}(\rho^\lambda u^\lambda) + \mathbf{H}(\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)) \\ &= \mathbf{H}(\operatorname{div}(\rho D(u^\lambda) + \operatorname{div}(\lambda E^\lambda \otimes \lambda E^\lambda) + (\rho^\lambda - 1) \nabla \Delta \rho^\lambda)). \end{aligned} \quad (83)$$

By using together (49) and the Proposition 7.2 for any $\varphi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$ we obtain that

$$\langle \partial_t \mathbf{H}(\rho^\lambda u^\lambda) + \mathbf{H} \operatorname{div}(\lambda E^\lambda \otimes \lambda E^\lambda), \varphi \rangle \longrightarrow \langle \partial_t \mathbf{H} u, \varphi \rangle. \quad (84)$$

The convergence established in (35) entails that for any $\varphi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$

$$\begin{aligned} \langle \mathbf{H}((\rho^\lambda - 1) \nabla \Delta \rho^\lambda), \varphi \rangle &= - \langle \nabla(\rho^\lambda - 1) \Delta \rho^\lambda, \mathbf{H} \varphi \rangle \\ &\quad - \langle (\rho^\lambda - 1) \Delta \rho^\lambda, \nabla \mathbf{H} \varphi \rangle \longrightarrow 0. \end{aligned} \quad (85)$$

The convergence of the diffusive terms follows in the following way.

$$\begin{aligned} \langle \mathbf{H} \operatorname{div}(\rho^\lambda D(u^\lambda)), \varphi \rangle &= - \langle \rho^\lambda u^\lambda, D(\nabla \mathbf{H} \varphi) \rangle + \langle \nabla \rho^\lambda \cdot u^\lambda, \nabla \mathbf{H} \varphi \rangle \\ &= - \langle \rho^\lambda u^\lambda, D(\nabla \mathbf{H} \varphi) \rangle + 2 \langle \sqrt{\rho^\lambda} u^\lambda \nabla \sqrt{\rho^\lambda}, \nabla \mathbf{H} \varphi \rangle \\ &\longrightarrow \langle \mathbf{H}(\Delta u^\lambda), \varphi \rangle, \end{aligned} \quad (86)$$

where we used (38), (49) and (66). For the convergence of the convective term it is enough to notice that by (i) and (iii) we have that $\sqrt{\rho^\lambda}$ and $\rho^\lambda u^\lambda$ converges almost everywhere hence

$$\sqrt{\rho^\lambda} u^\lambda = \frac{\rho^\lambda u^\lambda}{\sqrt{\rho^\lambda}} \longrightarrow u \quad \text{almost everywhere.} \quad (87)$$

And, as a consequence

$$\langle \mathbf{H} \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda), \varphi \rangle \longrightarrow \langle \mathbf{H} \operatorname{div}(u \otimes u), \varphi \rangle \quad (88)$$

So, by using together (84), (85), (86), (88) we have that $u = \mathbf{H}u$ satisfies the following equation in $\mathcal{D}'([0, T] \times \mathbb{R}^3)$

$$\mathbf{H}(\partial_t u - \Delta u + (u \cdot \nabla)u) = 0.$$

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